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Introduction.

1. The Ausdehnungslehre of Grassmann [1] has been applied to problems in geometry in two ways. In the form of vector analysis it has been used in solving problems of a metrical type. In the form of point analysis (with homogeneous coördinates as a basis) it has been used in problems of a descriptive nature. In projective geometry both of these methods have certain advantages and also certain disadvantages. The values of distances and angles occurring in vector analysis are useful as variables in terms of which to express projective relations. Yet the fact that these quantities are invariant under Euclidean motion has no place in projective geometry. The ternary form of the point algebra is of great value, but it is a decided disadvantage that λA and A (where A is a point) though distinct are not descriptively distinguishable. It is our aim in this paper to show how this coefficient λ can be interpreted as an angle invariant under a group of motions determined by the algebra itself and thus while using the homogeneous form retain the essential advantages of the metrical system.

We first develop the two systems of analysis in a purely projective way. Assuming that points and lines are represented by letters and that addition follows the usual laws, we find that there must exist exceptional elements somewhere in the plane. In fact, expressions λA, where A is a point and λ infinite, are not subject to the laws of addition. In the language of coördinate geometry such points correspond to infinite values of the coördinates and hence lie on a line. Likewise the lines λa where a is a line and λ infinite are exceptional lines passing through a point. Thus in such a system of algebra there exists a fixed point and a fixed line. We define our additions relative to these. In the vector addition the sum of two points A and B is a point C such that the harmonic of the singular line with respect to C and the fixed point is the harmonic of the same line with respect to A and B. This addition is characterized

[1] Gesammelte Werke, Vol. I, 1896.

by the fact that λA is in general distinct from A. In the point addition the sum of two points A and B is the harmonic of the singular line with respect to those points. For this addition λA is in general coincident in position with A. The two additions are closely related. In fact, each is representable in terms of addition processes of the other kind. When the singular line is taken at infinity these additions agree essentially with those of Grassmann.

2. In the case of point addition A and λA have the same position. According to Möbius these quantities differ in weight. To give a geometric interpretation to this weight we conceive a point as a sort of double fan-shaped spread consisting of all the lines through the point and between two limiting lines. The size of the point is then measured by the angle (directed) between these limiting lines, and the weight of the point is this angle. We are thus led to define a species of angle in which the total angular magnitude about a point is infinite. The finite angle determined by two lines is that one which does not contain the singular point.

In the same way we represent a line by a segment of itself and the magnitude of the line by the length of that segment. We thus define a sort of distance in which the locus of point at a fixed distance from a given point is a straight line. This distance between two points is the dual of the angle between two lines. It has a definite algebraic sign and along each line not passing through the singular point assigns a definite positive direction.

Distance and angle are invariant under a three-parameter group of collineations (projectively equivalent to motions leaving area invariant) for which the singular point and line are fixed elements. These collineations leave invariant the correlations having the fixed point and line as coincidence loci. There are two cases depending on whether the fixed point is on the line or not. The first of these gives a distance theory similar to that in a minimum plane. The second does not occur as a special case of distance defined relative to a conic.

3. In terms of this linear distance and angle there is a very simple theory of the triangle. Most rational relations of ordinary trigonometry involving distances and sines of angles are replaced by similar relations involving distances and angles. In case the singular point is on the singular line there is a linear relation between the sides and also between the angles of a triangle making both distance and angle similar to angle in ordinary geometry. If the point is not on the line, however, any three parts determine the triangle. Every part is then rationally expressible in terms of any three, and similar triangles do not exist.

In this system there is no right angle, and distance from point to line

does not properly exist. There is, however, a number associated with a point and line which has some of the properties of distance from point to line. We define an area that has the usual sum properties and in fact becomes identical (after proper choice of unit) with Euclidean area when the singular line is thrown to infinity.

In the course of the work we use the notations AB and ABC for segment and area respectively. Interpreting these as products we find that they have the properties of Grassmann's products. They are definable in terms of our distance, angle and area, in the same way that outer products are expressible in metrical concepts of Euclidean geometry.

The method used in this paper is not postulational. In fact, we are more interested in the results than in the method of obtaining them. In some cases our definitions have not been as simple as possible. In accordance with our primary aim we have interpreted quantities necessarily existing in the algebra instead of introducing notions that were not required.

4. It is our purpose to use this scheme in the solution of problems in projective geometry. Two ways of doing this are suggested. In the first place the above scheme of distance and angle gives us a great variety of coördinate systems. We may, for example, represent a point by its distance from two fixed points (singular point not on singular line) and a line by the angles it makes with two fixed lines. The equations of point and line are then of first degree and the incidence relation bilinear. The distance between two points is a bilinear function of their coördinates. Similarly for the angle between two lines. The differential of arc is of the form

$$xdy - ydx.$$

A kind of curvature is easily defined and thus we build up a differential projective geometry of plane curves.

In the second place we may start with the theory of the triangle. In terms of our distance, angle and area, we can then express descriptive relations such as perspectivity, inscribability in a conic, etc., and by processes similar to those in Euclidean geometry determine from these their projective consequences. This is especially easy since the expressions determining one part of a triangle in terms of three others are all rational. It is our intention to develop these applications in a later paper.

§ 1. Addition.

5. The addition of two points $\mathbf{A}$, $\mathbf{B}$ naturally divides itself into two cases depending on the significance given to $\lambda\mathbf{A}$. In metric geometry these two additions have been defined in terms of metric concepts. If

$\lambda \mathbf{A}$[2] is a point different in position from **A**, the addition familiar to all is the well known vector addition. If $\lambda \mathbf{A}$ is a point coincident in position with **A**, but differing from it in magnitude, the sum of two points is commonly defined as the centroid of the two points. We shall here define these two additions in terms of projective notions.

Vector Addition.

6. To define the sum of two points **A**, **B** when λ times a point differs in position from it, we assume a line f as reference line and a point O as origin. Then the sum $\mathbf{A} + \mathbf{B}$ is the point **C** determined as follows:

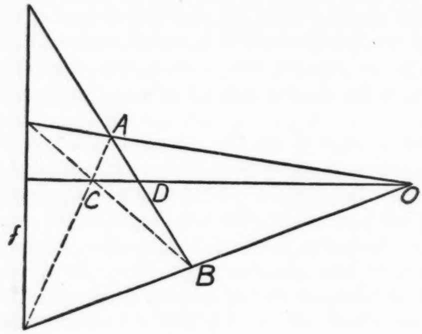

FIGURE 1.

Let D be the harmonic of f with respect to A, B. Then take a point C so that D is the harmonic of f with respect to C, O. The point C thus obtained is the point required, and we express the relation between the points by the equation

$$\mathbf{A} + \mathbf{B} = \mathbf{C}. \tag{1}$$

The geometric construction for C is shown in Figure 1, and is seen to be the ordinary vector construction where the line f has replaced the line at infinity.

From the above definition the point

$$\mathbf{A} + \mathbf{A} = 2\mathbf{A}$$

[2] In this paper capitals will be used for points, small letters for lines. Points or lines occurring as algebraic quantities will be represented by clarendon type, their magnitudes by italics. Geometric points or lines may, however, be represented by italics when no ambiguity results. Greek letters will be used for numbers.

is the point on the line OA such that A is the harmonic of f with respect to 2A, O. The point

$$2\mathbf{A} + \mathbf{A} = 3\mathbf{A}$$

from the definition is constructed as follows: Let C be the harmonic of f with respect to 2A, A and then construct the point 3A so that C is the harmonic of f with respect to 3A, O. It is at once seen from the construction of 2A that C is the point $\frac{3A}{2}$. From the theory of cross ratios it is also seen that the relation between 3A, A is expressed by the cross ratio

$$(O, f \mid 3A, A) = 3.$$

In like manner the points $\lambda\mathbf{A}$ for integral values of λ are constructed. If λ is the reciprocal of an integer μ, the point $\lambda\mathbf{A}$ can be constructed thus: Take a point B not on the line OA and construct the points

$$\mathbf{B},\ 2\mathbf{B},\ 3\mathbf{B}, \ldots \ldots \mu\mathbf{B}.$$

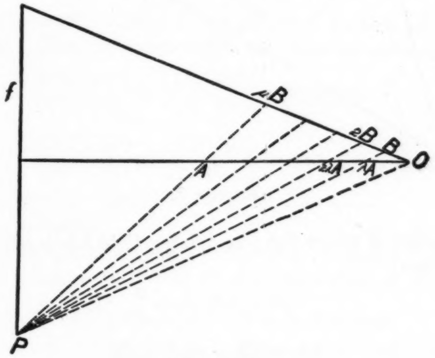

FIGURE 2.

Connect $\mu\mathbf{B}$ to $\mathbf{A}$ and let the line cut f in P. Draw a line from the point B to P. Where PB cuts OA is the point required, for the same harmonic relations hold among the A's as among the B's. The construction of the points $\lambda\mathbf{A}$ for rational values of λ is now evident. For irrational values of λ the construction is obtained by limiting processes. From the theory of cross ratios it is seen that the relation between $\mathbf{A}$ and $\lambda\mathbf{A}$ is expressed by the cross ratio

$$(O,f\,|\,\lambda A,\, A) = \lambda,$$

and every point on the line OA can be represented in the form $\lambda\mathbf{A}$.

7. The point

$$\lambda\mathbf{A} + \mu\mathbf{B},$$

where A and B are not collinear with O, is constructed in the same way as the point $\mathbf{A} + \mathbf{B}$, and has the following relations to the points A and B.

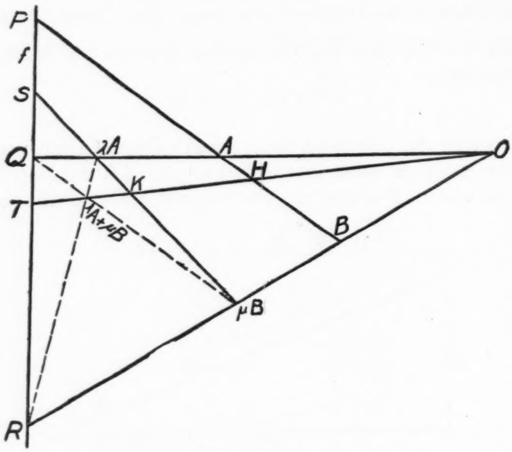

FIGURE 3.

(1) The point H where the line joining O to $\lambda\mathbf{A} + \mu\mathbf{B}$ cuts the line AB is such that

$$(Hf\,|\,AB) = -\frac{\mu}{\lambda}$$

$$\lambda\mathbf{A} + \mu\mathbf{B} = (\lambda + \mu)\,\mathbf{H}. \tag{2}$$

In order to prove the above relations we shall first prove the following lemma:

If two lines intersecting in O and cutting f in Q and R are cut by two other lines (intersecting f in P, S) in A, C and B, D respectively, and if through O any arbitrary line is drawn cutting AB in H, CD in K, and f in T, then

$$\frac{(OQ\,|\,AC)}{(OR\,|\,BD)} = \frac{(HP\,|\,AB)}{(KS\,|\,CD)}.$$

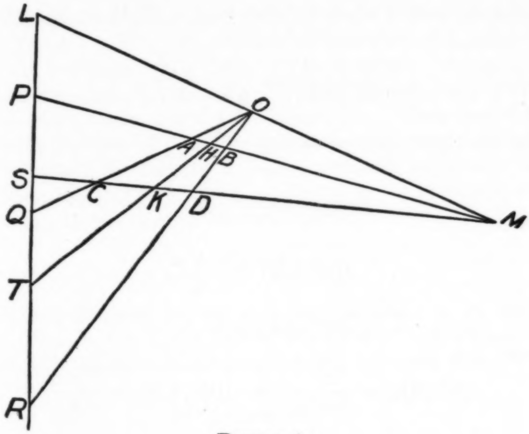

FIGURE 4.

Let M be the intersection of AB and CD, and project all points on f from O and M. The two ratios above can then be written as follows:

$$\frac{(OQ\,|\,AC)}{(OR\,|\,BD)} = \frac{(LQ\,|\,PS)}{(LR\,|\,PS)} = (RQ\,|\,PS)$$

$$\frac{(HP\,|\,AB)}{(KS\,|\,CD)} = \frac{(TP\,|\,QR)}{(TS\,|\,QR)} = (SP\,|\,QR);$$

but
$$(RQ\,|\,PS) = (SP\,|\,QR),$$
which proves the lemma.

If we apply the above lemma to Figure 3, replacing C by λA, D by μB, and K by $\frac{1}{2}(\lambda A + \mu B)$, it is at once seen that H divides AB in the cross ratio $-\frac{\mu}{\lambda}$ with respect to f. For, in this case,

$$(KS\,|\,CD) = -1,$$

$$(OQ\,|\,AC) = \frac{1}{\lambda},$$

$$(OR\,|\,BD) = \frac{1}{\mu}.$$

From which it follows that

$$(HP\,|\,AB) = -\frac{\mu}{\lambda}.$$

This shows that H is independent of the position of O.

To prove the second relation connecting A, B, H we have from the lemma

$$\frac{(\mathrm{OQ}\,|\,\mathrm{A}\;\lambda\mathrm{A})}{(\mathrm{OT}\,|\,\mathrm{HK})} = \frac{(\mathrm{BP}\,|\,\mathrm{AH})}{(\mu\mathrm{BS}\,|\,\lambda\mathrm{A}\;\mathrm{K})},$$

and, from the theory of cross ratios,

$$(\mathrm{OQ}\,|\,\mathrm{A}\;\lambda\mathrm{A}) = \frac{1}{\lambda},$$

$$(\mathrm{BP}\,|\,\mathrm{AH}) = \frac{\lambda + \mu}{2},$$

$$(\mu\mathrm{BS}\,|\,\lambda\mathrm{AK}) = 2.$$

Therefore

$$(\mathrm{OT}\,|\,\mathrm{HK}) = \frac{2}{\lambda + \mu} \quad \text{or} \quad (\mathrm{OT}\,|\,\mathrm{KH}) = \frac{\lambda + \mu}{2}.$$

This last relation is equivalent to the statement that

$$\mathbf{K} = \frac{\lambda + \mu}{2}\mathbf{H}.$$

But

$$\mathbf{K} = \tfrac{1}{2}\,(\lambda\mathbf{A} + \mu\mathbf{B}).$$

Therefore

$$\lambda\mathbf{A} + \mu\mathbf{B} = (\lambda + \mu)\,\mathbf{H}. \tag{2}$$

8. From the construction of $\mathbf{A} + \mathbf{B}$ the sum is seen at once to obey the following algebraic laws:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

$$\lambda\mathbf{A} + \mu\mathbf{A} = (\lambda + \mu)\,\mathbf{A}.$$

If $\quad \mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C} \quad \text{then} \quad \mathbf{B} = \mathbf{C}.$

The associative law

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

evidently holds for points on a line passing through O. To prove that it holds for any three points $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, project them on a line drawn through O from two different points P, Q on f. Let the projections be $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$ and $\mathbf{A}''$, $\mathbf{B}''$, $\mathbf{C}''$. The sum $\mathbf{A} + \mathbf{B} + \mathbf{C}$ will project into $\mathbf{A}' + \mathbf{B}' + \mathbf{C}'$ and $\mathbf{A}'' + \mathbf{B}'' + \mathbf{C}''$. The sum $\mathbf{A} + \mathbf{B} + \mathbf{C}$ will then be

at the intersection of the line joining P to $\mathbf{A}' + \mathbf{B}' + \mathbf{C}'$ with the line joining Q to $\mathbf{A}'' + \mathbf{B}'' + \mathbf{C}''$, which is independent of the way in which **A**, **B**, **C** are combined.

9. For the dual of the above addition we assume a point O′ as reference point, and a line f′ as reference line. Then from duality we see that the lines $\lambda\mathbf{a}$ all pass through the same point on f′. The sum $\mathbf{a} + \mathbf{b}$ is formed by drawing a line c harmonic of O′ with respect to a, b. Then drawing the line d so that c is the harmonic of O′ with respect to f′, D, the relation between c, d is

$$\mathbf{d} = 2\,\mathbf{c}.$$

The relations between λa, μb, a, b are the same as in the case of points. That is, the line h joining the point of intersection of f′ and $\lambda\mathbf{a} + \mu\mathbf{b}$ with the point of intersection of a, b is such that

$$(\mathrm{hO}' \,|\, \mathrm{ab}) = -\frac{\mu}{\lambda} \tag{1}$$

$$\lambda\mathbf{a} + \mu\mathbf{b} = (\lambda + \mu)\,\mathbf{h}. \tag{2}$$

The reference elements O′, f′ need not be the same as O, f. However, if the addition of points and its dual are to be used together, the line joining λA, λB should be λ times the line joining A, B. The line joining λA to λB intersects the line AB on f, for λA and λB (for variable λ) are two projective ranges with O as self-corresponding point, and consequently the ranges are perspective. The line f joins corresponding points (corresponding to the infinite value of λ) and therefore lines joining pairs of corresponding points must pass through the same point on f. Then if the line joining λA, λB is to be λ times AB the line f′ must coincide with f. The dual argument will show that O′ should coincide with O. The fundamental or reference system for this vector addition then consists of a line f and a point O.

The line f is exceptional in the addition for there is no way shown of finding the sum of two distinct points on this line.

Point Addition.

10. In this case $\lambda\mathbf{A}$ [3] is coincident in position with **A** but differs from **A** in magnitude. The number λ indicates the magnitude of $\lambda\mathbf{A}$. In the case of vector addition we saw that all points on a line through O

[3] Throughout the discussion of point addition points denoted by **A**, **B**, **C**, ... will always be understood to be unit points.

can be represented as multiples of one of them. Then if an arbitrary line AB, not passing through O, is drawn in the plane, all points on a line through O can be expressed as multiples of the point in which this line cuts AB. Now project all points of the plane on the line AB. Each point in the plane will be uniquely represented on AB if we use the following convention for magnitude. Let K be any point and let OK intersect AB in C. If $\mathbf{K} = \lambda \mathbf{C}$ we shall say that the projection on AB of the point K is the point λC. That is, the point in which K projects is considered as having a magnitude λ and is coincident in position with C. We thus see that all points on OK will project into the same point C, but the projection of each point will be looked upon as having a different magnitude. Now to find the sum $\lambda \mathbf{A} + \mu \mathbf{B}$ we will consider $\lambda \mathbf{A}$ and $\mu \mathbf{B}$ as represented vectorially and find the vector sum, then project the three point $\lambda \mathbf{A}$, $\mu \mathbf{B}$, $\lambda \mathbf{A} + \mu \mathbf{B}$ on AB. The projections will have the magnitudes λ, μ, $\lambda + \mu$ (see equation (2)) respectively. If the point $\lambda \mathbf{A} + \mu \mathbf{B}$ projects into $(\lambda + \mu)$ $\mathbf{C}$ then for points on the line AB we shall say that

$$\lambda \mathbf{A} + \mu \mathbf{B} = (\lambda + \mu)\, \mathbf{C}.$$

From equation (2) the point C is such that

$$(\mathrm{Cf} \mid \mathrm{AB}) = -\frac{\mu}{\lambda}.$$

The definition of this sum is then independent of the origin chosen for the vector addition. The reference element for this addition consists of the line f and we have: *The sum of two points* $\lambda \mathbf{A}$, $\mu \mathbf{B}$ *of magnitude* λ, μ *respectively is a point* $(\lambda + \mu)$ $\mathbf{C}$ *of magnitude* $\lambda + \mu$, *dividing* A, B *in the cross ratio* $-\frac{\mu}{\lambda}$ with respect to f.

The line f is a line of exceptional points. The sum of two distinct points on this line is not defined and the sum of two coincident points on f may be any given point of the plane, and therefore violates the uniqueness of the sum; besides, λ times a point P on f is not necessarily a point coincident in position with P.

The magnitude of the sum was seen to be the sum of the magnitudes. The difference

$$\mathbf{A} - \mathbf{B}$$

will be a point P of magnitude zero such that

$$(\mathrm{Pf} \mid \mathrm{AB}) = 1,$$

and consequently P must be on f. This point P is an exceptional point then because it is of zero magnitude and yet is not algebraically equiva-

lent to zero. The line f plays the role of the line at infinity, and the point P of zero magnitude is analogous to zero times infinity.

11. By duality we obtain a definition for the sum of two lines $\lambda\mathbf{a}$, $\mu\mathbf{b}$ of magnitudes λ, μ respectively. For this addition we assume a fundamental point F and

$$\lambda\mathbf{a} + \mu\mathbf{b}$$

is a line $\mathbf{c}$ of magnitude $\lambda + \mu$ such that

$$(\mathrm{CF}\,|\,\mathrm{ab}) = -\frac{\mu}{\lambda}.$$

The point F is an exceptional point, i. e. all the lines passing through F are exceptional in the same sense in which the points on f were exceptional. The difference $\mathbf{a} - \mathbf{b}$ is the line of zero magnitude joining the point F to the intersection of a and b. Here also the line of zero magnitude passing through F is not algebraically equivalent to zero but indeterminate. It is analogous to zero times infinity. The complete fundamental system then consists of a line f and a point F.

12. Starting with the definition of point addition the vector addition could be derived from it. Thus take a fixed point $\mathbf{O}$ and represent any point A in the plane by the difference $\mathbf{O} - \mathbf{A}$. The addition of these quantities will lead exactly to the vector addition with which we started. The two additions are then related in such a way that either can be derived from the other.

From the definition of point addition as derived from the vector addition it follows that the point addition obeys the same algebraic laws as the vector addition.

Vectorially any point in the plane can be expressed in terms of two independent points $\mathbf{A}$, $\mathbf{B}$, where the line AB does not pass through O. Then choosing λ, μ properly any point in the plane can be represented by $\lambda\mathbf{A} + \mu\mathbf{B}$. For the point addition, however, $\lambda\mathbf{A} + \mu\mathbf{B}$ represents only the points of the line AB, where neither A nor B is on f. In this addition, however, any point $\mathbf{X}$ of the plane can be expressed as

$$(\lambda + \mu + \nu)\,\mathbf{X} = \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are three non-collinear fixed unit points not on f. That is, the numbers λ, μ, ν can be so determined that this relation holds for any point whatever of the plane. To show this connect X to B. This line will cut AC in Q, which can be expressed in terms of $\mathbf{A}$ and $\mathbf{C}$. Then $\mathbf{X}$ can be expressed in terms of $\mathbf{Q}$ and $\mathbf{B}$. Thus

$$(\lambda + \nu)\,\mathbf{Q} = \lambda\mathbf{A} + \iota\mathbf{C}$$

$$(\lambda + \mu + \nu)\,\mathbf{X} = (\lambda + \nu)\,\mathbf{Q} + \mu\mathbf{B} = \lambda\mathbf{A} + \mu\mathbf{B} + \nu\mathbf{C}.$$

Since **A**, **B**, **C** are linearly independent, if

$$\lambda + \mu + \nu = 0$$

then X is on f.

From the above equations

$$(\lambda + \mu + \nu)\,\mathbf{X} - \mu\mathbf{B} = (\lambda + \nu)\,\mathbf{Q}.$$

Hence

$$(\mathrm{QP}\,|\,\mathrm{XB}) = \frac{\mu}{\lambda + \mu + \nu}$$

where P is the point in which BQ cuts f. Similar relations involving λ, ν can be obtained. Since λ, μ, ν are proportional to uniquely determined cross ratios the expression for X is unique.

§ 2. Distance and Angle.

13. We have considered two kinds of addition, either of which is expressible in terms of the other. The vector addition gives for the points of the plane a non-homogeneous, two-dimensional representation, the point addition, a homogeneous three-dimensional. The latter is more satisfactory for descriptive problems and will be assumed as the fundamental addition throughout the remainder of this paper. It has been remarked that expressions of the form **B** − **A**, when **A** and **B** are unit points, then combine according to the vector addition. Through a study of their expressions we shall derive a sort of distance that is intimately connected with the subject under discussion.

14. Vectors. We have seen that **B** − **A** represents a point of zero magnitude on f. We might infer that, if

$$\mathbf{B} - \mathbf{A} = \mathbf{D} - \mathbf{C},$$

the lines AB and CD pass through a common point on f. This is proved by writing the equality in the form

$$\mathbf{A} + \mathbf{D} = \mathbf{B} + \mathbf{C}.$$

The line f thus has the same harmonic E with respect to A, D and B, C. Let AD and BC cut f in P and Q. Then from the intersection of AB and CD the points A, E, D, P and B, E, C, Q are perspective. Since f joins two corresponding points, it must pass through the intersection of AB and CD. Writing the equality in the form

$$\mathbf{A} - \mathbf{C} = \mathbf{B} - \mathbf{D}$$

we see also that AC cuts BD on f. Thus the relation of the four points is shown by the diagram (Figure 5).

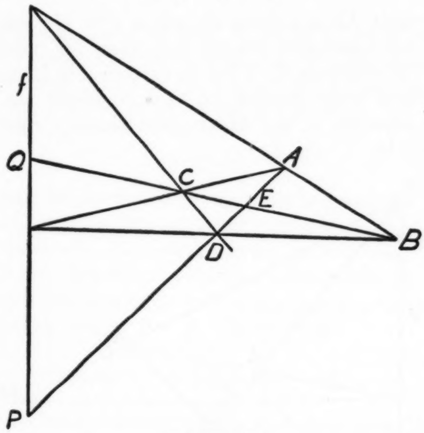

FIGURE 5.

Conversely, if the points A, B, C, D have the positions shown in the figure, f has the same harmonic with respect to A, D and B, C, or

$$\mathbf{A} + \mathbf{D} = \mathbf{B} + \mathbf{C}.$$

Consequently,

$$\mathbf{B} - \mathbf{A} = \mathbf{D} - \mathbf{C}.$$

If the four points lie on a line cutting f in P, we project, from some other point on f, the points A, B to M, N on a second line through P. Then from the intersection of MC and f we project N to D′ on AB. By construction

$$\mathbf{B} - \mathbf{A} = \mathbf{N} - \mathbf{M} = \mathbf{D}' - \mathbf{C}$$

Hence, if

$$\mathbf{B} - \mathbf{A} = \mathbf{D} - \mathbf{C},$$

the points $\mathbf{D}$ and $\mathbf{D}'$ must coincide. The relation of the four points is shown in Figure 6.

The quantity $\mathbf{B} - \mathbf{A}$ has properties very similar to those of the vector AB in the ordinary vector analysis. In fact, the expressions $\mathbf{B} - \mathbf{A}$ add exactly like vectors if the line f is at infinity. On account of this similarity we shall use the term vector to indicate the quantity $\mathbf{B} - \mathbf{A}$.

If

$$\mathbf{B} - \mathbf{A} = \lambda\,(\mathbf{D} - \mathbf{C}) \tag{3}$$

we might say that AB is λ times as long as CD. In what follows we shall assume this relation of lengths only when the points A, B, C, D lie on a line. In that case A, B can be expressed as linear functions of C, D, those points being assumed to be distinct and not on f. Since the sum of coefficients in any linear point identity must be zero, the expressions will be

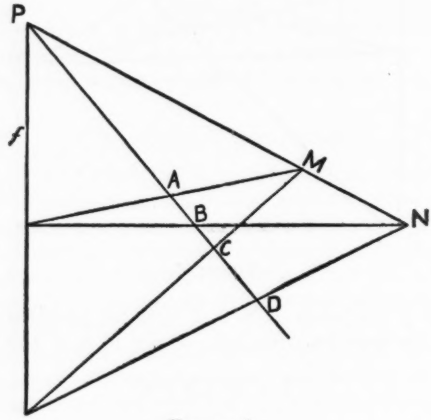

FIGURE 6.

$$\mathbf{A} = \lambda_1\,\mathbf{C} + (1 - \lambda_1)\,\mathbf{D},$$
$$\mathbf{B} = \lambda_2\,\mathbf{C} + (1 - \lambda_2)\,\mathbf{D}.$$

Then

$$\mathbf{B} - \mathbf{A} = (\lambda_1 - \lambda_2)\,(\mathbf{D} - \mathbf{C}).$$

Hence

$$\lambda = \lambda_1 - \lambda_2.$$

Now from (2) we have

$$-\frac{1 - \lambda_1}{\lambda_1} = (\mathrm{AP}\,|\,\mathrm{CD})$$

where P is the intersection of CD with f. Hence from the properties of double ratios

$$\lambda_1 = (\mathrm{AC}\,|\,\mathrm{DP}),$$
$$\lambda_2 = (\mathrm{BC}\,|\,\mathrm{DP}).$$

And consequently

$$\lambda = (AC \mid DP) - (BC \mid DP). \tag{4}$$

It is to be observed that λ is finite for all positions of A and B distinct from P, but becomes infinite when one of these points coincides with P. One of the points A or B being fixed, there is an unique position of the other which gives λ a particular value.

Distance.

15. In this way we determine the relation between the magnitudes of vectors on lines intersecting on f, but arrive at no relation between vectors not so situated. To obtain a more general relation we define distance as a scalar quantity, or number, determined by two points not on f and such that distances $\overline{AB}$[4] along any line are proportional to the corresponding vectors $\mathbf{B} - \mathbf{A}$. In symbols

$$\text{distance } AB \equiv \overline{AB} = K(\mathbf{B} - \mathbf{A})$$

when K is a constant for pairs of points A, B on a given line but may (and usually does) change from line to line.

It is not obvious that there exists a distance satisfying the above definition. We shall first show (if it exists) what such a distance must be. We shall find that it is not unique and then shall make a further assumption of a function theoretic nature. Finally we prove that the distance found has the properties required.

From the definition we see that for points on a line, if

$$\mathbf{B} - \mathbf{A} = \lambda\,(\mathbf{D} - \mathbf{C}),$$
$$\overline{AB} = \lambda\overline{CD}.$$

From the latter equation follows the former provided $\overline{AB}$ is not zero. In particular

$$\overline{AB} = -\,\overline{BA} \tag{5}$$

showing that distance is directed. Putting in this equation $\mathbf{B} = \mathbf{A}$, it follows that

$$\overline{AA} = 0. \tag{6}$$

Furthermore, since

$$\mathbf{B} - \mathbf{A} + \mathbf{C} - \mathbf{B} = \mathbf{C} - \mathbf{A},$$

---

[4] The notation $\overline{AB}$ will be used to denote the distance from A to B.

for collinear points

$$\overline{AB} + \overline{BC} = \overline{AC}. \tag{7}$$

16. Distance on a line. The definition of distance can be satisfied for points *on a given line* in two ways.

(1) There may exist two points C, D on the line such that $\overline{CD}$ is finite and not zero. For every number λ and point A of CD there exists an unique point B on CD such that

$$\mathbf{A} - \mathbf{B} = \lambda\,(\mathbf{C} - \mathbf{D}).$$

Hence there exists an unique point on the line at any given distance from the point A.

(2) There may exist two distinct points C, D on the line such that $\overline{CD}$ equals zero. In this case the above equation shows that the distance between any two points on the line CD (but not on f) is zero. For the point P in which CD cuts f, λ is infinite. Hence

$$\overline{AP} = \infty \cdot 0$$

is indeterminate. We assume that this distance may have any value whatever. Here again we find an unique point at a distance not zero from A.

Along most lines our distance will be of the first type. Along certain lines, however (analogous to minimal lines in metric geometry), distance will be of the second type.

17. Distance in the plane. Along each line through a point A is a single point at a given distance from A. There is a certain locus of these points. We assume that this locus is an analytic curve. Since it cuts each line through A in a single point it is then a straight line. Thus the locus of points at a given distance from a given point is a straight line.

We assume that there exist distances $\overline{AB}$ not zero. Let

$$\overline{AB} = k$$

where k is a constant not zero and A, B points not on f. If A is held fixed B describes a line b cutting f in P. If B is held fixed A describes a line a cutting f in a point Q. Distance along any line through A except AP is of type (1). Along AP it is of type (2). Similarly distance along every line through B except BQ is of type (1). Let the intersection of AP and BQ be F. Through any point A_1 except F there passes a line A_1A or A_1B along which distance is of type (1). Therefore, for any such point A_1 there exists a line of points b, such

that $\overline{A_1B_1} = k$. The only line through A_1 for which distance is of type (2) is the line joining A_1 to the intersection of b_1 and f. The intersection of two such lines must be exceptional, and since F is the only exception it follows that along any line through F the distance between two points not on f is zero and conversely if the distance between two non-coincident points is zero, their joining line passes through F.

We have just shown that if in the relation

$$\overline{AB} = k$$

A is held fixed, B describes a line b cutting f in a point P on the line AF. To determine more exactly this relation between A and b we consider the pairs of points A, B along a line CD not passing through F. On this line A and B satisfy an equation

$$\mathbf{B} - \mathbf{A} = \mathbf{D} - \mathbf{C}$$

where C, D is a particular pair. From the construction for equal vectors it is seen that A, B are corresponding points of a collineation on CD for which P is the only double point. In general to each point A corresponds a line b and for fixed B, A lies on a line. Hence to points A on a line CD correspond lines b through a point. Furthermore, these lines b pass through points B which are projective on CD with A. The correspondence between A and b is therefore a correlation. Since A and B coincide only on f, that line is the locus of points lying on their corresponding lines. To a point P on f corresponds a line through P. The distance from P to any point of this line being finite the distance between ordinary points on the line must be zero. Hence the line passes through F. This is true whether we hold A or B fixed on f. Hence F and f are corresponding elements in the correlation.

The preceding results may be summed up in the statement that if $\overline{AB}$ and $\overline{CD}$ are equal, B and D lie on the correspondents of A and C with respect to a correlation in which F and f are corresponding elements and f the coincidence locus. The construction of these correlations depends on whether F is or is not on f.

18. Point F on line f. We first consider the case in which the point F is on the line f. We have seen that the locus of points at a given distance from a point A is a line through the intersection of FA with f. In the present case all such lines pass through F. The correlation that determines equal distance is degenerate with F as singular point. The distance from this point to any point whatever is indeterminate.

Take two lines AC and BD passing through F. Since all points of BD are at the same distance from A and all points of AC at the same distance from D, and $\overline{AB} = -\overline{BA}$,

$$\overline{AB} = \overline{AD} = \overline{CD}.$$

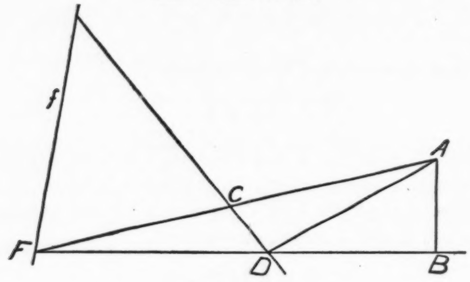

FIGURE 7.

Therefore the distance from any point of a line through F to any point of another line through F is constant. The distance between two points depends only on the lines joining them to F.

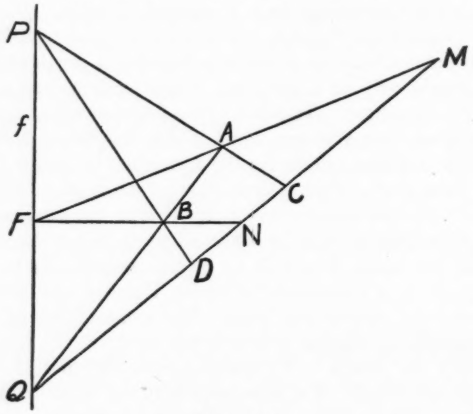

FIGURE 8.

Take any line CD, not passing through F, as a scale. Let distances along this line be proportional to the corresponding vectors. Then any distance AB is equal to the distance CD into which it projects

from F. The ratio of two vectors on a line is a difference of double ratios involving the end points of the vectors and the point of intersection of their line with f. That ratio is then unchanged when we project from F upon another line. Therefore distances as here constructed are proportional along any line to the corresponding vectors and consequently satisfy our definition.

In this case equal vectors are always of equal length, i. e. the opposite sides of a parallelogram are equal (parallel lines intersecting on f). This is shown in Figure 8. Let ABCD be a parallelogram. Draw FA and FB to cut CD in M and N. Then since the vectors MN and CD are equal,

$$\overline{AB} = \overline{MN} = \overline{CD}.$$

We shall see later that if F is not on f, equal vectors on different lines are not in general of equal length.

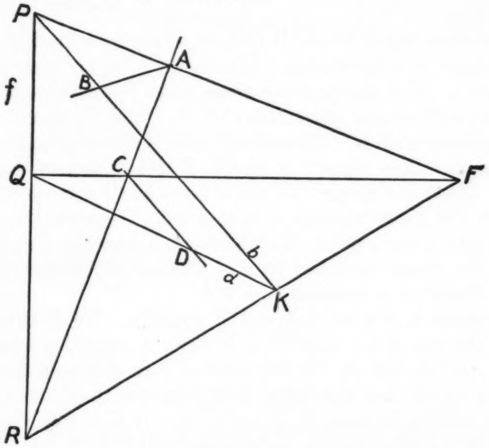

FIGURE 9.

19. Point F not on line f. If F is not on f, the locus of points at a given distance from a point A is a line through the point P in which FA cuts f. The locus of points at a constant distance from P is the line FP. The equation

$$\overline{AB} = k$$

gives for fixed A a line b, the locus of points B, and b is the correspondent of A in a certain correlation. If a certain point C and its

corresponding line d are given, we construct b the correspondent of A as follows. Draw AF and CF cutting f in P and Q and let CA cut f in R. To R corresponds the line FR. Then since three points A, C, R on a line must have corresponding to them three lines through a point, b, d, and FR pass through a point. Let d cut FR in K. Then PK is the line b required. The construction is shown in Figure 9.

To construct on any line through A a distance $\overline{AB}$ equal to a distance $\overline{CD}$, we construct the line b and where it cuts the line through A is the point B required. If on a fixed line CD we determine a scale making distances along that line proportional to the corresponding vectors $D - C$, our construction enables us to transfer the scale to any other line except f and so to assign to every pair of points, not on f, a distance. We must still show that this construction is consistent (if two distances so constructed are equal to a third, they are equal to each other) and that the resulting distance has the properties assumed in the definition.

We first show that if C and D (Figure 9) are held fixed, A and b are correspondents in a correlation. The figure gives for each point A an unique line b. If A moves along a line, since F and C are fixed, P and R describe, on f, ranges projective with A. Also K describes on d a range projective with R. Therefore P and K describe on f and d projective ranges. Also when A is on CF, P and R, and consequently K, are at Q. Since the ranges on f and d have a self-corresponding point Q, the line PK passes through a fixed point. Thus as A moves along a line, b turns about a point. Furthermore, b describes a pencil projective with the range described by A. The correspondence between A and b is therefore a correlation.

In particular if A is on f, A and R coincide. To R therefore corresponds the line RF. Also if A is at C, b coincides with d. The fact that Ab, Cd, and R, FR are pairs of corresponding elements in a correlation shows that the three lines intersect in K and thus determines the relation between A, B and C, D. If $\overline{A_1B_1}$ and $\overline{A_2B_2}$ are both equal to $\overline{CD}$, we have just shown that A_1, b_1 and A_2, b_2 are corresponding pairs in a correlation which gives for a point R on f the line RF. Therefore A_1, B_1 and A_2, B_2 are related by the same kind of diagram as A, B and C, D. Consequently, if two distances are by this construction equal to a third, they are equal to each other.

We now show that distances as constructed along any line AB are proportional to the corresponding vectors $\mathbf{B} - \mathbf{A}$. We have already remarked that vectors along a line preserve their ratios when projected upon another line from a point on f. If we hold A and C fixed (Figure 9) we may consider AB as resulting from CD by first projecting

from Q upon RF and then from P upon AB. Vectors on AB have therefore the same ratios as vectors on CD. Since distance has been defined as proportional to the vector along CD, it is proportional to the vector along any other line AB.

20. It has been mentioned that two vectors on different lines may be equal though their lengths are not equal. This is shown in Figure 10. Let AC and BD intersect on f at P and let AB, CD cut PF in M, N. The vectors AB and CD are equal. Suppose now

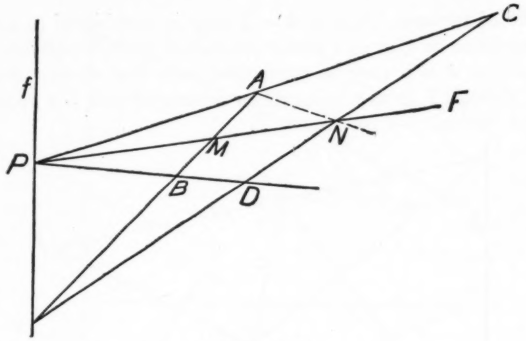

FIGURE 10.

$$\overline{AB} = \overline{CD}.$$

Then, by proportionality

$$\overline{AM} = \overline{CN},$$

and

$$\overline{BM} = \overline{DN}.$$

Also since $\overline{NA} = \overline{NC}$,

$$\overline{AM} = \overline{AN}.$$

Consequently P must be on the line AF. For the same reason B lies on the line FP. The vectors AB and CD are then zero. Non-vanishing equal vectors on different lines can not have the same length unless the point F lies on the line f.

21. Summary. In this section we have found a species of linear distance defined by a point F and line f. The distance between any two points on a line through F (but not on f) is zero. The distance between a point on a line through F and the point in which that line cuts f is indeterminate. The distance between two distinct points of a line not passing through F is finite and not zero but becomes infinite

when one of the points is on f. The locus of points at a constant distance from a point A is a line b through the intersection of FA with f. For fixed distance A and b are correspondents with respect to a definite correlation in which F and f are corresponding elements and f, the locus of points whose lines pass through them. On any line distances AB are proportional to the vectors $\mathbf{B} - \mathbf{A}$. Equal vectors on different lines have equal length if F is on f, but not otherwise.

Angle.

22. Point vectors. Just as $\mathbf{A} - \mathbf{B}$ may be considered as a point of zero magnitude on f, or as a vector associated with the segment AB not crossing f, so $\mathbf{a} - \mathbf{b}$ may be considered as a line of zero magnitude passing through F or as a point vector associated with the angular segment (or vector) not containing F.

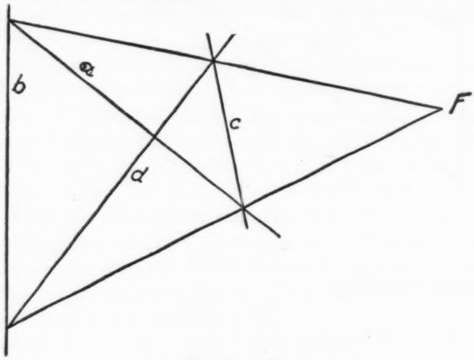

FIGURE 11.

Point vectors have properties dual to those of line vectors. Thus, if

$$\mathbf{a} - \mathbf{b} = \mathbf{c} - \mathbf{d}$$

ac and bd are points of a line through F. The relation of the lines a, b, c, d is shown in the diagram (Figure 11).

If
$$\mathbf{a} - \mathbf{b} = \lambda(\mathbf{c} - \mathbf{d})$$

where *c*, *d* are distinct and a, b, c, d pass through a point, we shall say the angle $\overline{ab}$ is λ times the angle $\overline{cd}$. In this case, if p is the line joining F to the common point,

$$\lambda = (ac \mid dp) - (bc \mid dp) \qquad (8)$$

and λ is finite for all positions of a and b distinct from p but becomes infinite when one of these lines coincides with p.

23. Definition and properties of angle. We now define angle as a scalar quantity determined by two lines not passing through F and such that pairs of lines through a point give angles proportional to the corresponding point vectors. We further assume that one of the lines being fixed and the angle kept constant, the locus of the other line is an analytic curve. That locus is then a straight line.

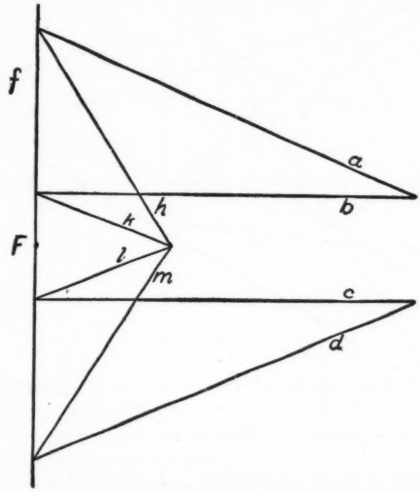

FIGURE 12.

Angle is thus the dual of distance. We find accordingly that there exists a line f such that if two lines (not passing through F) intersect on f, their angle is zero. If one of the lines passes through F, the angle is indeterminate. Two lines not intersecting on f determine an angle that is not zero. This angle is finite if neither of the lines passes through F, but infinite if one of them passes through that point. If

$$\overline{\mathrm{ab}} = k,$$

the locus of b for fixed a is a point B on the line joining fa to F. The correspondence between a and B is a correlation in which f and F are corresponding elements and F is the locus of lines passing through their corresponding points. There are two cases depending on whether F is

on f, or not. The constructions for equal angles are shown in Figures 12 and 13.

The equal angles are $\overline{ab}$ and $\overline{cd}$. In Figure 12,

$$\mathbf{h} - \mathbf{k} = \mathbf{l} - \mathbf{m}.$$

24. In the discussion of distance we found besides the fixed line f (locus of exceptional points in our algebra) a fixed point F′, that need

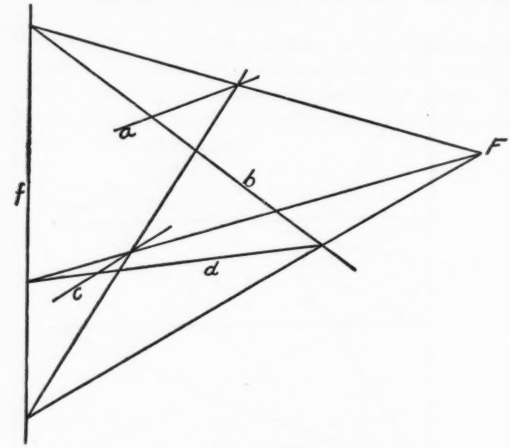

FIGURE 13.

not be the fixed point of our algebra. Again in the discussion of angle we have a fixed point F (locus of exceptional lines) and a fixed line f′, which need not be the fixed line occurring in the algebra. If we transform the plane by a collineation leaving F′, and f fixed, the ratios of distances will not be changed. If then there are to exist fixed relations between distances and angles in a figure (between sides and angles of a triangle, for example) the ratios of angles must be at the same time unchanged. We therefore choose the same fixed elements for distance and angle. These must then be the exceptional elements F and f in our algebra.

Distance as here discussed has a definite algebraic sign and hence along every line is assigned a definite positive direction. In Euclidean geometry there is no definite direction along a line because by a rotation preserving distance one direction can be turned into the other. That it is not possible, by our construction for equal distances, to rotate one

direction along the line into the other is shown in Figure 14. Rotation of AB around A causes B to describe the line BP. We have defined AB to be the segment not crossing f. Hence, as B passes P, the segment AB changes into the segment AB′ not crossing f (i. e. connecting A to B′ by way of infinity). Thus all segments into which AB can be rotated lie on the same side of AF and consequently one end of the line AB cannot be rotated into the other.

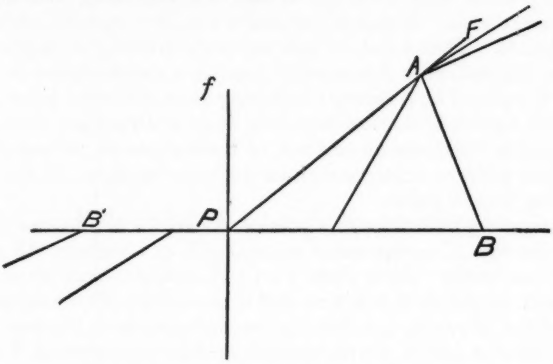

FIGURE 14.

A point describing (from A to B) any one of the segments AB (Figure 14) determines the same direction of rotation about F. Distances are then positive or negative according as their description gives rotation in one direction or the other around F.

Similarly the angle ABP is the angle which does not include F. If A is held fixed and B moves along BP, the angle is not changed. As B passes P the angle changes into AB′P. If around B we rotate the side AB into the side BP, the direction of translation (of the point of intersection of AB with f) along f is always the same. Hence an angle is positive or negative according as this translation along f is in one direction or the other.

In Euclidean geometry we have an unique direction of angle about any point since the equation

$$\text{angle} = \text{const.},$$

one side being fixed, factors into two linear conditions and so gives a separation of classes, but no definite direction along a line since

$$\text{distance} = \text{const.},$$

one point being fixed, is an irreducible quadratic form. Our distance and angle are more like Euclidean angle than Euclidean distance.

Metric representation of points and lines.

25. We wish to obtain a representation of point and line which distinguishes A and λA in the case of point addition. Then λA is coincident in position with A but differs from it in something that we have called magnitude. Grassman replaced a line by a segment joining two of its points. Then $\mathbf{a}$ and $\lambda\mathbf{a}$ give segments differing in length. We wish in this section to determine, if possible, a representation in which a line is replaced by a segment beginning at an arbitrary point of the line, and a point by a sector beginning at an arbitrary line through the point and to determine an addition of these segments or sectors such that their addition relations shall be the same as those of the corresponding lines or points.

Consider the lines through a point A not on f. We assume that to a line through A, corresponds a segment **AB**, to a segment **AB** a point B, and conversely. Since there is a [1,1] correspondence between the lines and the points B, it follows that to an addition of lines corresponds an addition of points, and that the two additions have the same formal laws. Since $\mathbf{a}$ and $\lambda\mathbf{a}$ are represented by different segments, B and λB are different points. The addition of points thus suggests the vector addition of § 1. The segments corresponding to lines through a point should then add like vectors from that point as origin. Distances along a line being proportional to the corresponding vectors, $\mathbf{a}$ and $\lambda\mathbf{a}$ should be represented at A by segments whose lengths have the ratio λ.

We thus represent a line of magnitude λ by a segment of length λ joining two points A, B of the line. Unit lines are represented by segments of unit length. These segments are like localized vectors in physics. Segments on the same line are proportional to their lengths. Segments on different lines may be added by moving them to the point of intersection of the two lines and there adding them like vectors.

To add two lines $\lambda\mathbf{c}$, $\mu\mathbf{d}$, of magnitudes λ and μ, we construct, at their point of intersection A, segments of lengths λ, μ ending in the points C, D. We then draw through A and the harmonic of f with respect to C, D the line h required. Our earlier method of constructing the sum $\lambda\mathbf{c} + \mu\mathbf{d}$ was to draw a line h through A such that

$$(\mathrm{hd}\,|\,\mathrm{cF}) = -\frac{\mu}{\lambda}.$$

That the two constructions are consistent follows since F and P (Fig-

ure 15) determine the same cross ratios with triads of lines through A. In fact, let

$$\overline{AB_1} = \overline{AB_2} = 1.$$

It was shown in § 1 that

$$\lambda\,(AB_1) + \mu\,(AB_2)$$

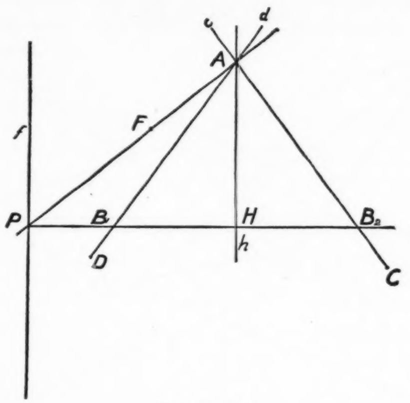

FIGURE 15.

intersects B_1B_2 in a point H such that

$$(HB_1 \mid B_2P) = -\frac{\mu}{\lambda}.$$

26. To add two segments we have moved them to the point of intersection of their lines. If the two lines intersect on f, this is not possible. We shall now derive a construction for the sum of two segments not at the same point. We have seen that equal vectors along a line (also equal segments on a line) project from a point on f into equal vectors on any other line. Since the sum of segments at a point is determined by harmonic constructions involving only those segments and f, it follows that addition relations connecting segments through a point are projective from a point on f. Any sum of segments may be found by a succession of processes consisting of moving two of the segments to the point of intersection of their lines and then adding them vectorially. Hence any linear relation connecting segments holds for the projections of those segments upon any line from a point on f.

This gives us a means of constructing the sum of two segments **AB** and **CD**. Let DA and BC cut f in P and Q. The sum **AB** + **CD** lies on some line LM through the intersection of AB and CD. From P (Figure 16) **AB** + **CD** is projected upon any line into a segment limited by the lines PC, PB. Likewise from Q it projects into a segment limited by QA, QD. Let PC intersect QA in L, and PB intersect QD in M. Then

$$\mathbf{AB} + \mathbf{CD} = \mathbf{LM}.$$

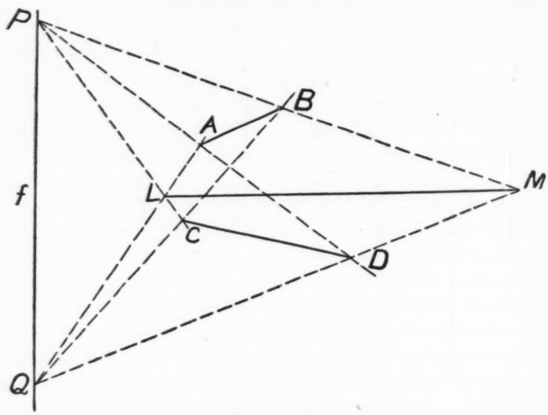

FIGURE 16.

27. Just as we represent a line by a segment (sequence of points between two given points) so we represent a point by a sector (sequence of lines between two given lines). The sector used is the one no line of which passes through F. A point of magnitude λ is represented by a sector of angle λ.

We conceive that the points in a discussion are thus replaced by fan-shaped spreads (the lines extending on both sides of the point). This spread, or sector, may be rotated about its vertex without changing its value by merely keeping the angle constant. Two of these sectors may be added vectorially after turning them around until the initial lines are the same. Let the sectors then be **ab** and **ac** and let the harmonic of F with respect to b and c be d. Then

$$\mathbf{ab} + \mathbf{ac} = 2\mathbf{ad}.$$

The sum of any two sectors may be found at once by a construction dual to that for adding segments (Figure 17).

We thus have two means of finding the sum

$$\lambda \mathbf{A} + \mu \mathbf{B}.$$

The one is an anharmonic construction involving the points A,B and the line f. The other is a harmonic or vector construction involving the sectors λA, μB and the point F. These constructions are both used in ordinary geometry, but the one only for adding points, the other only for adding lines. We have used both constructions in dual forms for both purposes.

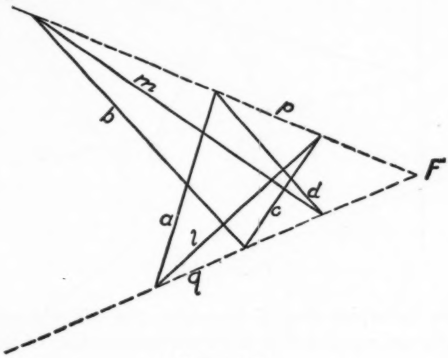

FIGURE 17.

§ 3. THE TRIANGLE.

28. In this section we shall apply the notions of distance and angle, as given in the previous section, to the study of plane figures, in particular to triangles. We have already seen that many properties are quite different according as the point F is on the line f or not. We shall therefore consider these two cases separately, taking first the case in which F is not on f and then the other as a special case.

Case I. *Point F not on line f.*

29. **Relation between distance and angle.** A comparison of the construction for equal distances with that for equal angles shows that the two are given by the same diagram. The group of collineations that leaves distance unaltered will then leave angle unaltered and conversely. We should therefore expect these quantities to be related.

When A is fixed and

$$\overline{\mathrm{AB}} = \text{const.}$$

B lies on a line b. The same figure gives

$$\overline{ab} = \text{const.}$$

where b is a fixed line and a is a line making a constant angle with b (Figure 18).

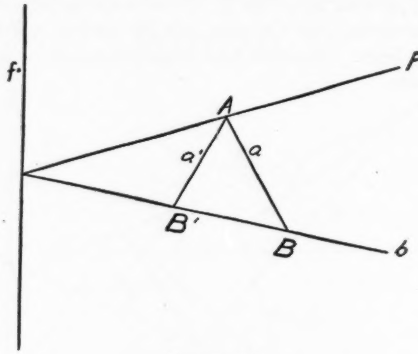

FIGURE 18.

This amounts to saying that if two sides of a triangle are equal the opposite angles are equal and have values independent of the third side or angle. Thus with every distance $\overline{AB}$ is associated an angle $\overline{ab}$. With any equal distance is associated the same or equal angle, since as previously mentioned the construction for equal distances at the same time makes the angles equal.

We shall determine the relation between these corresponding distances and angles. For this purpose take two distances $\overline{AB}$ and $\overline{AC}$ along the same line and with them the corresponding angles $\overline{ab}$ and $\overline{ac}$ (Figure 19). Let FA and AB cut f in P and Q. Draw FQ cutting PB and PC in D and E.

In the isosceles triangles ABD and ACE

$$\angle ABD\,[5] = \angle BAD,$$

[5] The angle ABD is the angle whose first side is AB and second side BD and which does not contain F. Thus in the figure the angle ABD is the angle ABP. The triangle ABD is isosceles in the sense that

$$\overline{PA} = \overline{DB}.$$

We shall later call triangles isosceles or equilateral when the sides described in a definite order around the triangle are equal, i. e. when

$$\overline{BD} = \overline{DA}.$$

$$\angle \mathrm{ACE} = \angle \mathrm{BAE}.$$

Hence
$$\frac{\overline{\mathrm{ab}}}{\overline{\mathrm{ac}}} = \frac{\angle \mathrm{ABD}}{\angle \mathrm{ACE}} = \frac{\angle \mathrm{BAD}}{\angle \mathrm{CAE}}.$$

From the definition of angle this last expression is the double ratio of the lines AD, AE, AB, AF, i. e. equal to

$$(\mathrm{DE} \mid \mathrm{QF}) = (\mathrm{BC} \mid \mathrm{AQ}) = (\mathrm{CB} \mid \mathrm{AQ}).$$

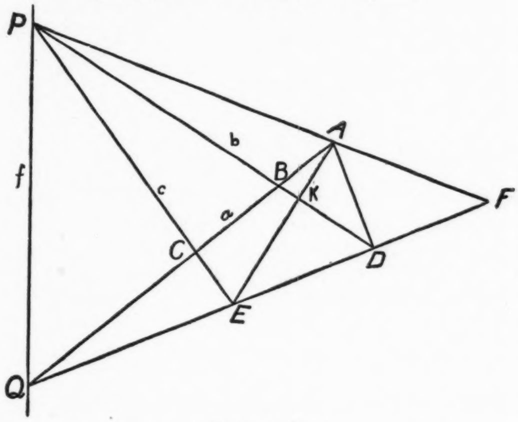

FIGURE 19.

From the definition of distance the last ratio is $\frac{\overline{\mathrm{AC}}}{\overline{\mathrm{AB}}}$. Hence

$$\frac{\overline{\mathrm{ab}}}{\overline{\mathrm{ac}}} = \frac{\overline{\mathrm{AC}}}{\overline{\mathrm{AB}}},$$

or
$$\overline{\mathrm{ab}} \cdot \overline{\mathrm{AB}} = \overline{\mathrm{ac}} \cdot \overline{\mathrm{AC}}.$$

We have already remarked that $\overline{\mathrm{ac}}$ is uniquely determined by $\overline{\mathrm{AC}}$. If then we keep $\overline{\mathrm{AC}}$ constant

$$\overline{\mathrm{ab}} \cdot \overline{\mathrm{AB}} = \mathrm{k}.$$

The product of side and angle in an isosceles triangle is an absolute constant. We shall choose the units so that the constant is unity.

Then
$$\overline{\mathrm{ab}} = \frac{1}{\overline{\mathrm{AB}}}. \tag{9}$$

Angle is therefore of dimension minus one in distance. In fact, since angle and distance are dual if there is to be a dimensional relation between them the first should be of the same dimensions in the second that the second is in the first. The dimensions could then be only 1 or —1 in distance.

We have $$\frac{\angle \mathrm{BAK}}{\angle \mathrm{BAD}} = (\mathrm{KD} \mid \mathrm{BP}) = \frac{\overline{\mathrm{BK}}}{\overline{\mathrm{BD}}}.$$

Now $$\angle \mathrm{BAD} = \frac{1}{\overline{\mathrm{AB}}}$$

and $$\overline{\mathrm{BD}} = \overline{\mathrm{AD}} = \overline{\mathrm{AB}}.$$

Hence $$\angle \mathrm{BAK} = \frac{\overline{\mathrm{BK}}}{\overline{\mathrm{AB}}^2}.$$

We may consider the line BD as a circle of radius $\overline{\mathrm{AB}}$ and center A. (The same line may be considered as a circle in an infinite number of ways but to each center corresponds a unique length of radius. Any point on AF may be taken as center of the same circle BD.) Then the angle at the center is given by

$$\text{angle} = \frac{\text{arc}}{(\text{radius})^2}. \tag{10}$$

30. Triangle relations. Analogy with Euclidean geometry leads us to expect relations between the sides and angles of a triangle. We shall now determine these relations.

Let ABC be the given triangle. Draw the lines as indicated in figure 20. Denote the length of the sides BC, CA, AB by a, b, c respectively and the angles CAB, ABC, BCA by A, B, C respectively.

Then $$a + b + c = \overline{\mathrm{BC}} + \overline{\mathrm{CA}} + \overline{\mathrm{AB}} = \overline{\mathrm{BE}} + \overline{\mathrm{DA}} + \overline{\mathrm{AB}} = \overline{\mathrm{DE}}.$$

Also $$\frac{\overline{\mathrm{DE}}}{\overline{\mathrm{DA}}} = (\mathrm{EA} \mid \mathrm{DR}) = (\mathrm{QS} \mid \mathrm{PR}) = (\mathrm{BH} \mid \mathrm{AR}) = \frac{\overline{\mathrm{AB}}}{\overline{\mathrm{AH}}}$$

Therefore $$A = \frac{1}{\overline{\mathrm{AH}}} = \frac{\overline{\mathrm{DE}}}{\overline{\mathrm{AB}} \cdot \overline{\mathrm{DA}}} = \frac{a+b+c}{bc}. \tag{11}$$

Similarly $$B = \frac{a+b+c}{ac}. \tag{12}$$

$$C = \frac{a+b+c}{ab}. \tag{13}$$

These relations solved for a, b, c give

$$a = \frac{A + B + C}{BC}, \tag{14}$$

$$b = \frac{A + B + C}{AC}, \tag{15}$$

$$c = \frac{A + B + C}{AB}, \tag{16}$$

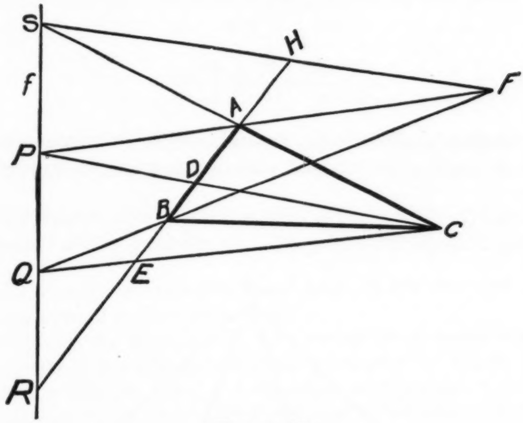

FIGURE 20.

which have the same form as before as they should have by duality. These equations can be solved for any three of the quantities A, B, C, a, b, c in terms of the other three. Thus any three of these parts determine the triangle uniquely. In particular the three angles determine the triangle and therefore similar triangles do not exist.

We can construct a triangle having any three given lengths for sides. In fact, take any segment BC of length a. Construct lines at distances c, b from B and C respectively. These lines intersect in a point A such that the triangle ABC has the sides required. There is no relation between the three sides of a triangle. Since any three parts determine the triangle there is no relation between any three parts. Since we have already found three independent equations connecting the six quantities it follows that any other must be a consequence of these.

It is to be noted in particular that the three angles of a triangle are not functionally related. The reason that the angles of a triangle in Euclidean geometry are so related is because angle is there of zero dimension in distance. The three angles of a triangle being homogeneous functions of zero dimensions in the sides are functions of two ratios $\frac{a}{c}, \frac{b}{c}$, and hence must be functionally related.

From (11) by division we get

$$\frac{A}{a} = \frac{a+b+c}{abc}.$$

Hence
$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}, \tag{17}$$

a set of relations similar to the sine proportions in trigonometry. In this system angle often replaces sine of the angle in ordinary trigonometry

31. Area. To determine the sides and angles of a triangle a direction around the triangle must be given. In speaking of the triangle ABC

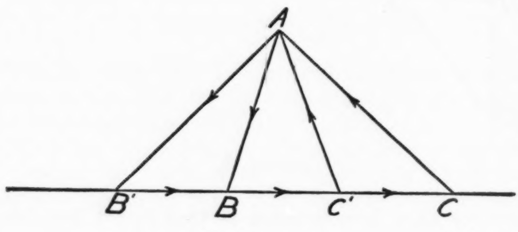

FIGURE 21.

we shall assume this direction to be A, B, C. The lengths of the sides are there $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ and the angles $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ where a is opposite A, b opposite B, etc. Quantities such as area connected with a triangle may depend on this direction of description.

We define the *line area* of a triangle as a scalar quantity, determined by three points taken in a definite order, such that triangles having a vertex in common and their bases on the same line have areas proportional to the lengths of their bases.

Take two triangles ABC and AB′C′ having the same vertex A and bases $\overline{BC}$, $\overline{B'C'}$ on the same line (Figure 21).

Consider the triangle AC′C. $\overline{\mathrm{AC}'} = -c'$, and angle AC′C $= -C'$, where C is the angle of the triangle AB′C′. Then from relation (17) we have

$$\frac{C'}{b} = \frac{C}{b'} \text{ or } b'C' = bC.$$

Then if we have the relation

$$\frac{\text{Area ABC}}{\text{Area A}'\text{B}'\text{C}'} = \frac{a}{a'},$$

multiplying both numerator and denominator by the above equality we have

$$\frac{\text{Area ABC}}{\text{Area A}'\text{B}'\text{C}'} = \frac{abC}{a'b'C'}.$$

The two terms of the right member are equal to the sum of the sides of the respective triangles. Hence

$$\frac{\text{Area A}'\text{B}'\text{C}'}{\text{Area ABC}} = \frac{a'}{a} = \frac{a' + b' + c'}{a + b + c}.$$

Triangles having the same vertex and bases on the same line have line areas proportional to their perimeters.

Starting with the triangle ABC by a succession of operations consisting of moving a side along its line and changing its length, we arrive finally at any triangle A′B′C′. Since under each of these operations the area is changed in the same ratio as the sum of the sides, it follows that the areas of any two triangles are proportional to their perimeters. We choose the unit of area such that

$$\text{Area ABC} = a + b + c. \tag{18}$$

That this expression for area has the properties required is evident since it is determined by an ordered sequence of three points, and since as already shown two triangles having the same vertex have perimeters proportional to their bases.

Perhaps the most fundamental property of area is the sum property, i. e. that the area of a region is the sum of the areas of its parts. This is a property of the sum $a + b + c$. For if we divide a triangle into two triangles by a line through one of the vertices the sum of the perimeters of the two is equal to that of the original triangle since the dividing line is counted twice in opposite directions. If then we define the

area of any closed polygon as the sum of the areas of the triangles into which it can be divided the area of any polygon will be its perimeter.

32. Similarly we define the *angle area* of a triangle as a scalar quantity determined by three lines taken in a definite order and such that triangles having the same vertex and bases on the same line have areas proportional to the angles at the vertex. By a proper choice of units the angle area takes the form

$$\text{Area } abc = A + B + C. \tag{19}$$

Defining the angle area in general as having the sum property we see that any closed polygon has an angle area equal to the sum of the angles between consecutive sides.

33. The formulae for the two areas may be written in other forms that show more clearly the analogy with the ordinary trigonometric formulae for area. Thus from the equation

$$A = \frac{a + b + c}{bc}$$

we have

$$\text{Area ABC} = a + b + c = bcA$$

analogous to the formula

$$2 \text{ Area ABC} = \text{bc} \sin \text{A}$$

in trigonometry. Our formula thus corresponds to the formula for twice the ordinary area.[6]

Again from the equation

$$a = \frac{A + B + C}{BC}$$

we have

$$\text{Area } abc = A + B + C = aBC$$

which is to be compared with the ordinary formula

$$2 \text{ Area } abc = a \sin \text{B} \sin \text{C}.$$

In both of these pairs of formulae for area angle replaces the sine of an angle of trigonometry.

[6] As the present scheme of distance and angle is entirely distinct from the ordinary distance and angle, the one being linear, the other quadratic, it does not seem advisable to complicate our formulae by the introduction of the multiplier ½ necessary to make the analogy complete. The only case common to the two systems is that in which the circular points at infinity coincide, which corresponds to our scheme when F is on f.

Another analogy is obtained by comparing the ordinary area of a sector of a circle with our corresponding area of an isosceles triangle. The area of sector of angle A and radius b is

$$\tfrac{1}{2}b^2\mathrm{A}.$$

The area of our isosceles triangle having a vertical angle A and the sides $\overline{\mathrm{AB}} = \overline{\mathrm{AC}} = b$ is

$$-\, b^2\mathrm{A}$$

which can be obtained from the formula Area $= bc\mathrm{A}$, remembering that in the isosceles triangle $\overline{\mathrm{AB}} = b$, $\overline{\mathrm{CA}} = c = -\,b$. Thus in this case angle replaces angle.

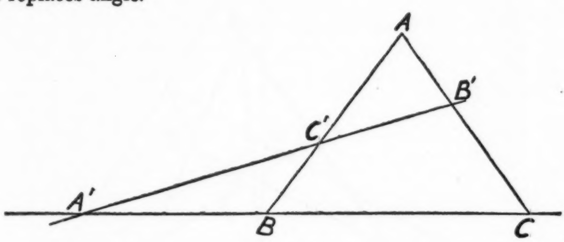

FIGURE 22.

34. In connection with the similarity between the trigonometric formulae in this geometry and in the ordinary geometry, it is interesting also to note some of the similarities and differences of the geometric relations of triangles in the two cases. The following theorems are good illustrations.

If the sides BC, CA, AB *of a triangle are cut by a transversal in the points A', B', C' respectively, then*

$$\frac{\overline{\mathrm{C'A}}}{\overline{\mathrm{C'B}}} \cdot \frac{\overline{\mathrm{A'B}}}{\overline{\mathrm{A'C}}} \cdot \frac{\overline{\mathrm{B'C}}}{\overline{\mathrm{B'A}}} = 1.$$

For writing the three vertices in order to denote the line area of the triangle we have (Figure 22)

$$\frac{\mathrm{A'B'C}}{\mathrm{C'B'A}} \cdot \frac{\mathrm{C'B'A}}{\mathrm{B'C'C}} \cdot \frac{\mathrm{B'C'C}}{\mathrm{C'A'C}} \cdot \frac{\mathrm{C'A'C}}{\mathrm{C'A'B}} \cdot \frac{\mathrm{C'A'B}}{\mathrm{A'C'A}} = 1.$$

Since triangles having the same vertex and bases on the same line have

line areas proportional to the lengths of their bases, this relation can be written

$$\frac{\overline{A'C'}}{\overline{C'B'}}\cdot\frac{\overline{B'A}}{\overline{CB'}}\cdot\frac{\overline{B'C'}}{\overline{C'A'}}\cdot\frac{\overline{A'C}}{\overline{A'B}}\cdot\frac{\overline{BC'}}{\overline{C'A}}=1,$$

which is equivalent to

$$\frac{\overline{C'A}}{\overline{C'B}}\cdot\frac{\overline{A'B}}{\overline{A'C}}\cdot\frac{\overline{BC}}{\overline{B'A}}=1.$$

The converse of this theorem is immediately seen to be true.

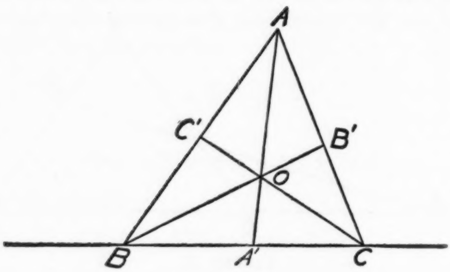

FIGURE 23.

If through any point O *in the plane of a triangle ABC lines are drawn to the vertices cutting the sides opposite* A, B, C *in* A′, B′, C′ *respectively, then*

$$\frac{\overline{A'C}}{\overline{A'B}}\cdot\frac{\overline{C'B}}{\overline{C'A}}\cdot\frac{\overline{B'A}}{\overline{B'C}}=-1.$$

For we have the identity (Figures 23 and 24),

$$\frac{\overline{OA'}}{\overline{OC}}\cdot\frac{\overline{OC}}{\overline{OB'}}\cdot\frac{\overline{OB'}}{\overline{OA}}\cdot\frac{\overline{OA}}{\overline{OC'}}\cdot\frac{\overline{OC'}}{\overline{OB}}\cdot\frac{\overline{OB}}{\overline{OA'}}=1,$$

which becomes, on rearranging the terms,

$$\frac{\overline{OA'}\cdot\overline{OC}}{\overline{OA}\cdot\overline{OC'}}\cdot\frac{\overline{OB'}\cdot\overline{OA}}{\overline{OA'}\cdot\overline{OB}}\cdot\frac{\overline{OC'}\cdot\overline{OB}}{\overline{OC}\cdot\overline{OB'}}=1,$$

or since triangles with the same vertical angle have areas proportional to the product of the including sides.

$$\frac{\mathrm{OA'C}}{\mathrm{OAC'}} \cdot \frac{\mathrm{OB'A}}{\mathrm{OBA'}} \cdot \frac{\mathrm{OC'B}}{\mathrm{OCB'}} = 1.$$

Rearranging

$$\frac{\mathrm{OA'C}}{\mathrm{OBA'}} \cdot \frac{\mathrm{OB'A}}{\mathrm{OCB'}} \cdot \frac{\mathrm{OC'B}}{\mathrm{OAC'}} = 1.$$

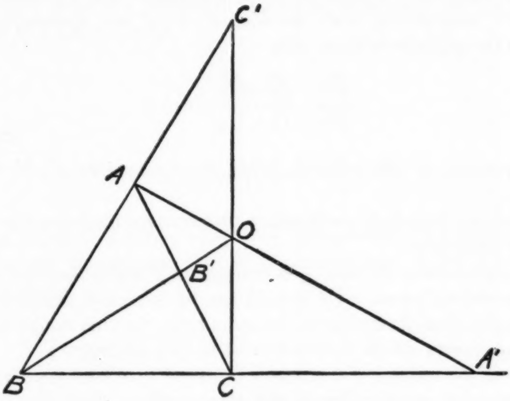

FIGURE 24.

But triangles having the same vertex and bases on the same line have line areas proportional to their bases. Hence

$$\frac{\mathrm{A'C}}{\mathrm{BA'}} \cdot \frac{\mathrm{B'A}}{\mathrm{CB'}} \cdot \frac{\mathrm{C'B}}{\mathrm{AC'}} = 1,$$

which on changing the sign of the denominators becomes

$$\frac{\overline{\mathrm{A'C}}}{\overline{\mathrm{AB'}}} \cdot \frac{\overline{\mathrm{B'A}}}{\overline{\mathrm{B'C}}} \cdot \frac{\overline{\mathrm{C'B}}}{\overline{\mathrm{C'A}}} = -1.$$

The converse of this theorem is also easily proved. These two theorems are the same for ordinary geometry as they are here, and in fact the demonstrations here given are applicable to either geometry.

In this geometry we have duals of the two preceding theorems, which is not the case in ordinary geometry.

If from a point in the plane of a triangle lines a', b', c' are drawn to the vertices A, B, C respectively, then

$$\frac{\overline{c'a}}{\overline{c'b}} \cdot \frac{\overline{a'b}}{\overline{a'c}} \cdot \frac{\overline{b'c}}{\overline{b'a}} = 1,$$

and conversely, *if the above relation holds, the three lines a', b', c' pass through the same point.*

If any line b in the plane of the triangle abc cuts the three sides in A', B', C', respectively, and the lines a', b', c' are drawn from these points to the opposite vertices, then

$$\frac{\overline{a'c}}{\overline{a'b}} \cdot \frac{\overline{c'b}}{\overline{c'a}} \cdot \frac{\overline{b'a}}{\overline{b'c}} = -1,$$

and conversely, *if this relation holds, the three points A', B', C' lie on a line.*

From these theorems result many propositions analogous to those of ordinary geometry:

Lines intersecting the base of a triangle on f divide the other two sides into proportional parts. (It should not be expected that the bases of the triangles thus formed have the same ratio, for this would be equivalent to symmetry which does not exist in this geometry.)

If one of the vertices of a triangle is joined to F and lines are drawn from any point on this line to the other vertices these lines divide the angles into proportional parts.

Lines drawn from the vertices to the mid points of the opposite sides of a triangle meet in a point which is a point of trisection for the medians, the longer part being toward the vertex.

The bisectors of the angles of a triangle meet the opposite sides in three points on a straight line. It should be observed that the three bisectors in this geometry cannot all cut the opposite sides between the vertices. This is because the angle was defined to be that one which does not contain F.

If through any point O in the plane of a triangle ABC, lines OA, OB, OC are drawn, intersecting the opposite sides in A', B', C'; and if $A'B'$, AB meet in G, $B'C'$, BC in H and $A'C'$, AC in K, the points G, H, K lie on a straight line.

35. Point-line Invariant. In this system we have nothing analogous to perpendicularity, and there is no minimum distance from point to line. If then we are to find a quantity analogous to the distance from a point to a line in Euclidean geometry the analogy must have as

its basis some other property. Now the distance from the vertex to the base of a triangle is ordinarily determined by the formula

$$\mathrm{h} = \mathrm{b} \sin \mathrm{C}.$$

Hence in our system we try the value

$$\delta = bC.$$

It is evident from the definition that δ depends only on the positions of A and BC, for we have

$$\frac{b}{B} = \frac{c}{C},$$

and hence

$$\delta = bC = cB = \frac{a + b + c}{a}. \tag{20}$$

The first value of δ shows that it is independent of the position of B along the line, the second, that it is independent of the position of C along the line.

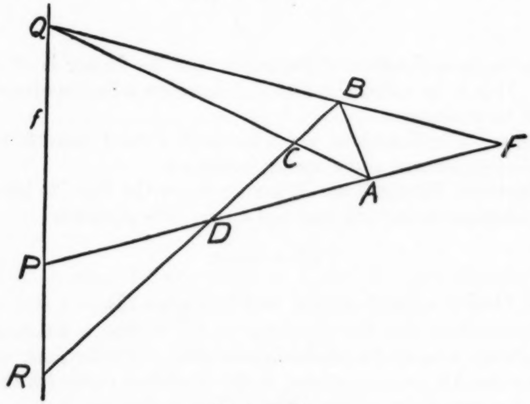

FIGURE 25

To determine more exactly the meaning of this quantity δ, we draw lines FA and FB meeting f in P and Q. Let BC cut AQ in C, FP in D, f in R.

From the triangle ABD we get

$$\delta = \frac{\overline{BD} + \overline{DA} + \overline{AB}}{\overline{BD}}.$$

Since in the isosceles triangle ABC, $\overline{AB} = \overline{CB} = c = -a$, and since $\overline{DA} = 0$, being measured on a line passing through F, the above relation can be written

$$\delta = \frac{\overline{BD} + c}{\overline{BD}} = \frac{\overline{BD} - a}{\overline{BD}} = \frac{\overline{CD}}{\overline{BD}} = (\mathrm{CB}\,|\,\mathrm{DR}) = (\mathrm{AF}\,|\,\mathrm{DP}).$$

Thus the invariant δ is the cross ratio determined by the point A, the point F and the points in which FA cuts the given line BC and the line f.

From the fundamental relations of the triangle we have

$$\frac{a+b+c}{a} = \frac{A+B+C}{A}.$$

Therefore

$$\delta = \frac{A+B+C}{A}, \tag{21}$$

which is the same function of the angles that the former is of the distances. It is to be noted also that if A describes a line passing through R, δ will be unchanged.

The group of collineations which leaves F, f fixed leave δ invariant although distance and angle are not invariant.

36. Metrical Illustration. When we throw the line f to infinity the system takes an interesting metrical form. The equation

$$\overline{AB} = \text{const.}$$

gives for fixed A a line b parallel to FA (Figure 26).

For points B on this line the distance $\overline{AB}$ is then a constant times the Euclidean area of the parallelogram ABF. Further, our distances along the line AB are proportional to the Euclidean distances and therefore to the area of the parallelogram. Hence distance from a point A is proportional to the area of the parallelogram ABF, or

$$\overline{AB} = k \,\square\, \mathrm{ABF}$$

where k is a function of A.

Similarly from B we have

$$\overline{BA} = k_1 \, \square \, BAF.$$

Now
$$\overline{BA} = -\overline{AB}$$

and
$$\square \, ABF = -\square \, BAF,$$

hence
$$k_1 = k,$$

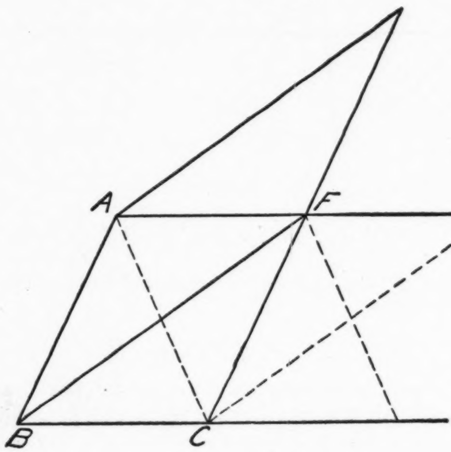

FIGURE 26.

and
$$\overline{AB} = k \, \square \, ABF$$

where k is constant for all values of A and B. We may choose the units so that k equals unity. Then

$$\overline{AB} = \square \, ABF. \tag{22}$$

We know that

$$\overline{ab} = \frac{\overline{BC}}{\overline{AB}^2},$$

consequently

$$\overline{ab} = \frac{\square \, BD}{(\square \, AC)^2} = \frac{1}{\square \, AC}. \tag{23}$$

Thus distance is the moment of Euclidean distance with respect to F and angle is the reciprocal of a moment.

It is to be observed that the sum of the three sides of a triangle is twice the Euclidean area. In fact,

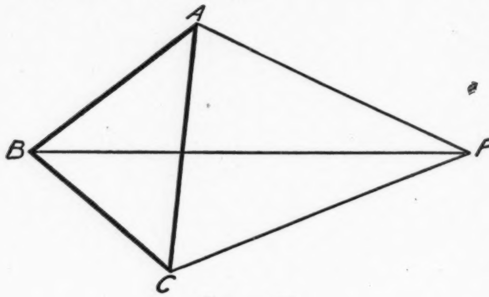

FIGURE 27.

$$\overline{AB} + \overline{BC} + \overline{CA} = 2\ (ABF + BCF + CAF) = 2\ ABC.$$

This is another justification for calling the sum of the sides of a triangle the area of the triangle.

CASE II. *The Point F on the Line f.*

37. In this case distances are equal if they project from F into equal vectors on a line. All triangles having vertices A, B, C respectively on fixed lines passing through F have equal sides, but may have different angles. Hence in this case it is not possible to express an angle in terms of the sides as was done in Case I.

Furthermore (Figure 28).

$$\overline{AB} + \overline{BC} = \overline{AD} + \overline{DC} = \overline{AC}.$$

Similar relations hold for angles. We thus have two equations connecting the parts of a triangle.

$$\begin{aligned} a + b + c &= 0, \\ A + B + C &= 0. \end{aligned} \tag{24}$$

We should expect one other relation. This may be found as follows: Let AB, BC, and CA cut f in P, Q, R. Then

$$\frac{a}{b} = \frac{\overline{DC}}{\overline{CA}} = -\ (DA|CR) = -\ (FP|QR).$$

Since angles are proportional to the point vectors determined at a fixed point by their intersections with the line f, we may consider them as represented by vectors on f measured relative to F as infinite point. Thus

$$\frac{A}{B} = \frac{\mathrm{RP}}{\mathrm{PQ}} = -\ (\mathrm{RQ}\,|\,\mathrm{PF}) = -\ (\mathrm{FP}\,|\,\mathrm{QR}).$$

Consequently

$$\frac{a}{b} = \frac{A}{B},$$

and

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}. \tag{25}$$

These equations, together with the two already found, form a set of three independent relations connecting the six parts of a triangle. A

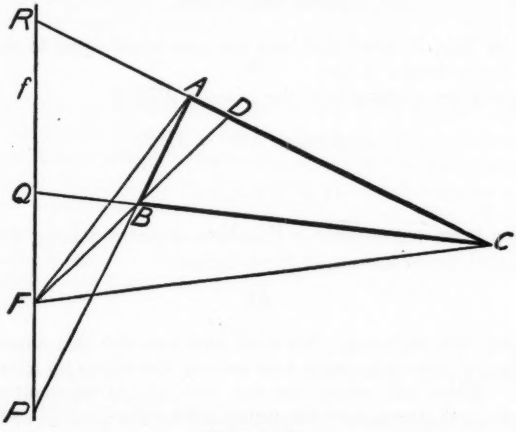

FIGURE 28.

triangle can be constructed having a given side and two given angles. There cannot then be more than three relations connecting the six parts, and therefore any other is a consequence of these.

38. Point-line Invariant. If we have a fixed point A and a fixed line LM and take a triangle with vertex A and base BC on the line LM, we have from the above relation

$$\delta = bC = cB, \tag{26}$$

which shows that the quantity δ is an invariant here just as in Case 1.

There is one difference, however, for in Case I this invariant was expressible as a ratio of distances, and consequently was independent of the unit of length assumed. In this case δ is not so expressible. It is not, therefore, an absolute invariant, but depends upon the units.

39. Area. The line area again will be defined by the property that triangles having the same vertex and bases on the same line have areas proportional to their bases. By the same argument as used in Case I it is seen that

$$\frac{\text{Area ABC}}{\text{Area A'B'C'}} = \frac{abC}{a'b'C'},$$

and the same argument will show that the factor of proportionality is the same for the whole plane and hence can be assumed to be unity, and we write

$$\text{Area ABC} = abC. \tag{27}$$

It is to be kept in mind that here the area is not equal to the perimeter, for the perimeter is zero.

By duality the angle area of the triangle ABC is

$$\text{Angle area ABC} = aBC. \tag{28}$$

§ 4. Products.

40. We have represented a line by a segment joining two of its points A and B and have used the notation

AB

to represent this segment. We shall now consider this expression as the product of two unit points and seek to determine its laws of combination. When the points are not unit points we shall take the product to be the segment[7] **AB** multiplied by the product of the magnitudes of the points.

The operation of multiplication will be denoted by writing the quantities together as is done in ordinary algebra. A dot between quantities will indicate how the quantities are to be associated, e. g. AB · CD will be used to indicate that the two products AB and CD are to be multiplied together.

[7] It should here be kept in mind that segment is not to be confused with its length, which measures only the magnitude of the segment. The length of a segment can be zero without the segment being zero, which is the case of a segment of a line passing through F.

From the fact that segments are directed we see that the product of two points obeys the alternative law

$$\mathbf{AB} = -\mathbf{BA}. \tag{29}$$

Also when the two points coincide the segment is zero, that is,

$$\mathbf{AA} = 0 \tag{30}$$

On the other hand, if

$$\mathbf{AB} = 0,$$

either the magnitude of one of the points is zero or the segment is zero. Then if neither point is a zero point and the product vanishes, the two points will coincide. Here again it is well to note that the product of two points on a line through F does not necessarily vanish although the segment is of zero length or magnitude.

41. We have used the notation

$$\mathbf{ABC}$$

to represent the triangle determined by the three points. We shall consider this also as the product of the three unit points and take it to mean the line area of the triangle determined by the three points. If the points are not unit points, the product will mean the area multiplied by the product of the magnitudes of the points. From this definition we see that the product vanishes if the three points are on a line, neither point being on f. Conversely, if the points are not zero points and the product vanishes, they are on a line. Therefore

$$\mathbf{ABC} = 0$$

is a necessary and sufficient condition that the three points be linearly related, i. e. collinear.

From the fact that area is directed we see at once that

$$\mathbf{ABC} = \mathbf{BCA} = \mathbf{CAB} = -\mathbf{ACB} = -\mathbf{CBA} = \ldots\ldots \tag{31}$$

42. From duality we now define the product of two lines as the sector determined by the two lines multiplied by the product of the magnitudes of the lines. When the two lines coincide the sector vanishes and we have as for points

$$\mathbf{ab} = 0.$$

Conversely, if the lines are not zero lines, $\mathbf{ab} = 0$ is the condition that the two lines should coincide. In the case of two lines as in the case

of two points it should be kept in mind that the magnitude of the sector may be zero without the product necessarily being zero, e. g. if the lines intersect on f. In this case the magnitude of the sector is zero, but the product is not zero. If we now take the two lines **a** and **b** as represented by segments starting at the point of intersection, thus

$$\mathbf{a} = \mathbf{BC} \qquad \mathbf{b} = \mathbf{AC}$$

where **A**, **B**, **C** are unit points, then

$$\mathbf{AB} \cdot \mathbf{AC} = (\mathbf{ABC})\,\mathbf{A}. \tag{32}$$

Since the magnitudes of the lines **a**, **b** are represented by the lengths of the segments, from the definition the product would be $\overline{BC} \cdot \overline{AC} \cdot \mathbf{A}_1$ where $\mathbf{A}_1$ represents the sector determined by the two lines. But $\mathbf{A}_1$ can be written as $A \cdot \mathbf{A}$ where **A** means a unit sector and the product takes the above form.

The product of three unit lines is defined to be the angle area of the triangle determined by the three lines, and the product of any three lines is this product multiplied by the product of the magnitudes of the lines. When the three lines pass through a point the angle area of the triangle is zero and conversely. Hence, *The necessary and sufficient condition that three lines* **a**, **b**, **c** (not zero lines) *should pass through the same point is*

$$\mathbf{abc} = 0.$$

Since the angle area is also a directed quantity

$$\mathbf{abc} = \mathbf{bca} = \mathbf{cab} = -\,\mathbf{bac} = \ldots\ldots \tag{33}$$

43. The product of a line **a** and a point **A** will be defined as the invariant δ of the point and line multiplied by the product of the magnitudes of the point and line. The invariant vanishes when and only when **A** lies on **a**. Therefore if neither **a** nor **A** are zero elements, *the necessary and sufficient condition that* **A** *lies on* **a** *is*

$$\mathbf{Aa} = 0.$$

If the line **a** is represented by a segment of length a determined by two unit points **B** and **C**, then from the above definition

$$\begin{aligned} \mathbf{Aa} &= aA\delta = aABc \\ &= a\,(ABc) \\ &= A\,(acB). \end{aligned} \tag{34}$$

That is, the product of a point and a line (represented by one of its segments) is equal to the line area of the triangle having the point for

vertex and the segment for base multiplied by the magnitude of the point. Or dually, the product is equal to the angle area of the triangle having the sector **A** for vertex and **a** for base line, multiplied by the magnitude of **a**. If the line a passes through F, the above product becomes indeterminate since $a = 0$ and $B = \infty$; this can be evaluated by

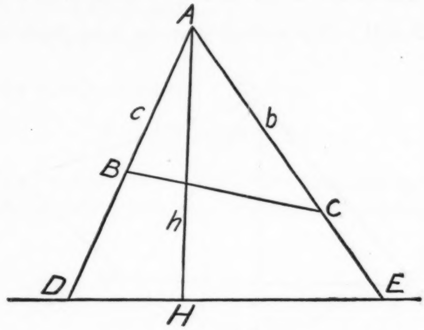

FIGURE 29.

remembering the relation $bA = aB$. Both b and A are finite and the product becomes determinate as before. From the above expression for the product we see that

$$\mathbf{Aa} = \mathbf{aA}. \tag{35}$$

44. The product of two unit points was defined as the segment joining the two points. This segment we may now consider as represented by the line joining the two points taken with a magnitude equal to the length of the segment. Then making use of the definition of the product of a point and a line we see at once

$$\mathbf{ABC} = \mathbf{AB} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{BC}. \tag{36}$$

That is, the product is associative. Likewise the product of three lines is associative.

45. The distributive law. The distributive law for points

$$(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \tag{37}$$

follows from the definition of addition. Let the magnitudes of the three points be λ, μ, ν respectively. Then (Figure 29)

$$\mathbf{AB} = \lambda\mu\mathbf{C}, \quad \mathbf{AC} = \lambda\nu\mathbf{b}.$$

Let $\overline{AD} = \lambda\mu c, \quad \overline{AE} = \lambda\nu b.$

Then by definition $\mathbf{AB} + \mathbf{AC} = 2\mathbf{h}$, where H is the harmonic of f with respect to D, E and h joins A to H. It was shown that the line AH divides the segment BC in the ratio $-\frac{\nu}{\mu}$ and therefore the line must pass through $\mathbf{B} + \mathbf{C}$. It is evident that the magnitude of $\mathbf{A}\ (\mathbf{B} + \mathbf{C})$

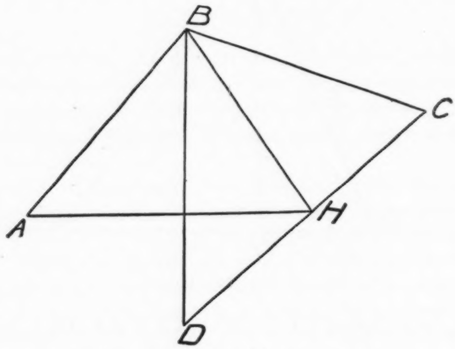

FIGURE 30.

is equal to the magnitude of $\mathbf{AB} + \mathbf{AC}$, each being equal to $\lambda\ (\mu + \nu)$, and since these lines have two points in common they are then equal.

The distributive law for points and lines

$$\mathbf{a}\ (\mathbf{C} + \mathbf{D}) = \mathbf{aC} + \mathbf{aD} \tag{38}$$

can be proved as follows : Let $\mathbf{a} = \mathbf{AB}$ where $\mathbf{A}$ and $\mathbf{B}$ are unit points. Then the left member can be written

$$\mathbf{AB}\ (\mathbf{C} + \mathbf{D}),$$

or, since the products are associative,

$$\mathbf{A}\ (\mathbf{BC} + \mathbf{BD}).$$

Let (Figure 30) $\mathbf{H} = \frac{1}{2}(\mathbf{C} + \mathbf{D}).$

Then $\mathbf{BH} = \frac{1}{2}\mathbf{B}\ (\mathbf{C} + \mathbf{D}) = \frac{1}{2}(\mathbf{BC} + \mathbf{BD})$

and
$$\begin{aligned}\mathbf{A}\,(\mathbf{BC}+\mathbf{BD}) &= 2\mathbf{ABH}\\ &= 2(\overline{\mathrm{AB}}+\overline{\mathrm{BH}}+\overline{\mathrm{HA}})\\ &= 2\overline{\mathrm{AB}}+(\overline{\mathrm{BC}}+\overline{\mathrm{BD}})+\overline{\mathrm{DA}}+\overline{\mathrm{CA}}\\ &= (\overline{\mathrm{AB}}+\overline{\mathrm{BC}}+\overline{\mathrm{CA}})+(\overline{\mathrm{AB}}+\overline{\mathrm{BD}}+\overline{\mathrm{DA}})\\ &= \mathbf{ABC}+\mathbf{ABD}\\ &= \mathbf{aC}+\mathbf{aD}.\end{aligned}$$

By duality the distributive law

$$\mathbf{A}\,(\mathbf{b}+\mathbf{c}) = \mathbf{Ab}+\mathbf{Ac} \tag{39}$$

is also true.

46. We have defined the product of two lines and have expressed it in terms of three unit points when the segments representing the two

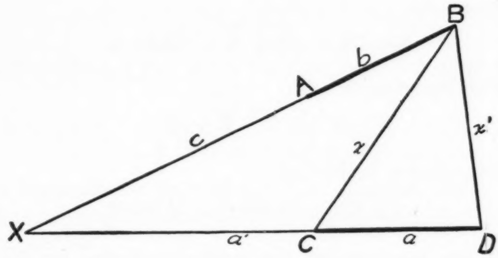

FIGURE 31.

lines have an end point in common. This product can also be expressed in terms of unit points when the segments representing the lines do not have an end point in common. That is, we know that the product $\mathbf{AB}\cdot\mathbf{CD}$ is the sector determined by the two lines and we seek to express this sector in terms of the four points $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. Grassmann has called such a product a regressive product. Let the two lines be represented by segments thus

$$\mathbf{a} = \mathbf{CD}, \quad \mathbf{b} = \mathbf{AB}$$

and let $\mathbf{X}$ be the unit point at the intersection of $\mathbf{a}, \mathbf{b}$.

Then we can express $\mathbf{A}$ in terms of $\mathbf{B}$ and $\mathbf{X}$.

$$(\lambda+1)\,\mathbf{A} = \mathbf{X}+\lambda\mathbf{B}. \tag{40}$$

Multiplying by **B**

$$(\lambda + 1)\,\mathbf{AB} = \mathbf{XB}.$$

Multiplying by **CD**

$$\begin{aligned}(\lambda + 1)\,\mathbf{AB}\cdot\mathbf{CD} &= \mathbf{XB}\cdot\mathbf{CD}\\ &= \mathbf{XB}\,(\mathbf{CX} + \mathbf{XD})\\ &= (\mathbf{XBD} - \mathbf{XBC})\,\mathbf{X}\\ &= (\mathbf{BCD})\,\mathbf{X}.\end{aligned}$$

Again multiplying (40) by **CD**

$$(\lambda + 1)\,\mathbf{ACD} = \lambda\,(\mathbf{BCD}).$$

Using the last two relations with (40)

$$(\lambda + 1)\,\mathbf{A} = \frac{(\lambda + 1)\,(\mathbf{ACD})}{(\mathbf{BCD})}\,\mathbf{B} + (\lambda + 1)\frac{\mathbf{AB}\cdot\mathbf{CD}}{(\mathbf{CBD})}.$$

From which we get

$$\mathbf{AB}\cdot\mathbf{CD} = (\mathbf{CDA})\,\mathbf{B} - (\mathbf{CDB})\,\mathbf{A}. \qquad (41)$$

Starting with the relation

$$(\lambda + 1)\,\mathbf{C} = \mathbf{X} + \lambda\mathbf{D}$$

the following relation is obtained :

$$\mathbf{AB}\cdot\mathbf{CD} = (\mathbf{ABD})\,\mathbf{C} - (\mathbf{ABC})\,\mathbf{D}. \qquad (42)$$

These last two relations are the general formulae for Grassmann's regressive product. Similar formulae can be obtained for the regressive product

$$\mathbf{ab}\cdot\mathbf{cd}.$$

Subtracting (41) from (42) we get

$$(\mathbf{ABC})\,\mathbf{D} - (\mathbf{ABD})\,\mathbf{C} + (\mathbf{ACD})\,\mathbf{B} - (\mathbf{BCD})\,\mathbf{A} = 0 \qquad (43)$$

for the identical relation connecting four points of the plane.

Scalar Products.

47. The point F was exceptional only in the addition of lines. It is lines through this point that are exceptional and not the point itself. The preceding product theory then holds when F is one of the quantities involved. If **A**, **B** and **F** are unit points not on f

$$(\mathbf{ABF}) = \overline{\mathrm{AB}} + \overline{\mathrm{BF}} + \overline{\mathrm{FA}}.$$

Hence

$$(\mathbf{ABF}) = \overline{\mathrm{AB}} \tag{44}$$

since the distances from F to A and B are zero. As F is fixed throughout the discussion we may consider (ABF) or $\overline{\mathrm{AB}}$ as a product of **A** and **B**. Since it is a number we call it the *scalar product*. If **A** and **B** are not unit points we shall use the notation $\overline{\mathrm{AB}}$ for the product (**ABF**). It is then the distance from A to B multiplied by the product of the magnitudes of **A** and **B**.

From the definition, if

$$\mathbf{B} + \mathbf{C} = \mathbf{D},$$

we have

$$\begin{aligned}\overline{\mathrm{AB}} + \overline{\mathrm{AC}} &= (\mathbf{ABF}) + (\mathbf{ACF}) \\ &= (\mathbf{ADF}) \\ &= \overline{\mathrm{AD}}.\end{aligned}$$

The scalar product of two points has then the following properties :

$$\overline{\mathrm{AB}} = -\,\overline{\mathrm{BA}}$$

$$\overline{\mathrm{A\,(B + C)}} = \overline{\mathrm{AB}} + \overline{\mathrm{AC}}. \tag{45}$$

These laws hold for both **AB** and $\overline{\mathrm{AB}}$. The two differ in the fact that **AB** is a multiple quantity, $\overline{\mathrm{AB}}$ a number. The first forms an associative product

$$\mathbf{AB} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{BC},$$

while the second does not.

If **a**, **b** and **f** are unit lines not passing through F,

$$(\mathbf{abf}) = \overline{\mathrm{ab}} + \overline{\mathrm{bf}} + \overline{\mathrm{fa}}.$$

Since the angles $\overline{\mathrm{bf}}$ and $\overline{\mathrm{fa}}$ are zero

$$(\mathbf{abf}) = \overline{\mathrm{ab}}. \tag{46}$$

The angle between two lines may then be considered as a *scalar product* of those lines. If **a** and **b** are not of unit magnitude, we define $\overline{\mathrm{ab}}$ by equation (45). This product dual to $\overline{\mathrm{AB}}$ has the following properties:

$$\overline{\mathrm{ab}} = -\,\overline{\mathrm{ba}} \tag{47}$$

$$\overline{\mathrm{a\,(b + c)}} = \overline{\mathrm{ab}} + \overline{\mathrm{ac}}.$$

Equations (45) together with the fact that $\overline{\mathrm{AB}}$ is a number suggests

geometry in one dimension. In fact, if a point A is represented as a linear function of three points, one of which is F, the multiple of F is lost in the product

$$\overline{AB} = (\mathbf{ABF}).$$

Thus, so far as concerns their values in products of the form $\overline{AB}$, points of the plane are linear functions of two points. In respect to this multiplication the plane is then one dimensional. Thus any homogeneous identical relation between products **AB** along a line will hold for distances $\overline{AB}$ in the plane. For example, along a line

$$\mathbf{AB} \cdot \mathbf{CD} = \mathbf{AC} \cdot \mathbf{BD} - \mathbf{AD} \cdot \mathbf{BC}.$$

Therefore

$$\overline{AB} \cdot \overline{CD} = \overline{AC} \cdot \overline{BD} - \overline{AD} \cdot \overline{BC} \tag{48}$$

where **A**, **B**, **C**, **D** are any four points in the plane. This can be proved directly by using the equation

$$(\mathbf{ABF})\ \mathbf{C} - (\mathbf{ABC})\ \mathbf{F} + (\mathbf{AFC})\ \mathbf{B} - (\mathbf{BFC})\ \mathbf{A} = 0$$

which is obtained from (43) by changing the notation. Multiplying by **DF** and transposing we get

$$(\mathbf{ABF})\ (\mathbf{CDF}) = (\mathbf{ACF})\ (\mathbf{BDF}) - (\mathbf{ADF})\ (\mathbf{BCF})$$

which is equivalent to (48).

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